Intermittency, Fractal Dynamics } Hurst Exponent, Anomalous Diffusion.

How relate/connect?

So Far:

- discussed degrees of randomness:
  - → mild (Gaussian) - CLT applies
  - → slow (lognormal)
  - * → wild ($\langle x^2 \rangle \to \infty$, i.e. Cauchy)
    CLT DOA

Challenge is dealing with 'wild' randomness; (Cauchy)

i.e. $P(x) = A / 1 + cx^2$

which is dominated by (very) fat tails.

⇒ Concentration in large events.

- explored multiplicative processes, lognormal distribution, stochastic ode, pde as intermittency models

⇒ lesson was importance of higher moments and γ_p scaling. HOM ⇔ concentration symptom!

- Fractal / β models
  - → spatial concentration
  - → non-uniformity of dissipation / dissipative structure
  - → defines new 'cartoon' → Koch curve as cartoon of real coastline ⇒ roughness

Begs (at least) two questions:

- symptom of fractality / intermittency in time? Nature of kinetics? How diagnose?
  - That t — probe time series of flux, other?
  - economic data time series
  - connection
- quantify, represent? ⇒ Hurst exponent, R/S etc.

- how reflect intermittency in transport calculation?

e.g. ⇒ Fokker-Planck Theory fails if $\langle \Delta x^2 P(\Delta x) \rangle \not< \infty$

so $P(\Delta x) = A / \left[B' + (\Delta x)^{\alpha+1} \right]$ (power law) — requires $\alpha > 2$ { $\alpha = 1 \rightarrow$ Cauchy

→ more generally, why represent by χ, D if transport fundamentally non-diffusive?!
e.g. Richardson $\delta l^2 \sim \epsilon t^3$

⇒ { Fractional Kinetics (next).
  Anomalous Diffusion.

N.B.: Technically, time series are "self-affine" (scaling) not "self-similar." → Linear operation that applies different reduction ratios along I, t axes, brings back to self.

~Classic example of a self-affine / scaling process is 1/f noise.

typical toppling power law (spike, $|\hat{\phi}_\omega|^2$)

So → Need characterize series.

How define "roughness" of series? ← diag.
"randomness"

⇒ Self-affine fractal models!

- Brownian motion, Bachelier 1900 { stock price increments as random walk }

B(t) ≡ 'random' process ⇒ time series (i.e. flux?) Gaussian increments

if

$$E\{B(t+T) - B(t)\} = 0,$$

↑ expectation

$$E\{(B(t+T) - B(t))^2\} = T$$

→ Fickian, Brownian motion
→ time series random, with short autocorrelation.
⇒ {orthogonality increments
  {with Gaussian → independent steps

So, now generalize:

have Fractional Brownian motion if:

for all t, T:

$$E\{B_H(t+T) - B_H(t)\} = 0$$

$$E\{(B_H(t+T) - B_H(t))^2\} = T^{2H}$$

$H \equiv$ Hurst; Holder exponent

H: $0 < H < 1$

$H = 1/2$; Fickian / Brownian → random (beware buzzwords)

$H > 1/2$; "Super-Diffusive" → ballistic / persistent dynamics

$H < 1/2$; "Sub-Diffusive" → sticky anti-persistent

Note that can re-express:

$$E\{(\Delta B^2)\} = (\Delta t)^{2H}$$

ΔB, Δt } finite increments

so

$$\ln|\Delta B| / \ln|\Delta t| \equiv H$$

coarse measurement

so obvious similarity:

$$D_0 = \ln N / \ln(1/\theta)$$

H is obviously related to:
- fractal dimension
- nature of series randomness.

Meaning of H? / Fractional Brownian Motion

→ model of diverse phenomena that exhibit cyclic, non-periodic variability at all scales.
characterizes variability

→ i.e. "Joseph effect" - movements / trends in time series tend to be part of larger trends/cycles rather than completely random. 'Joseph effect' refers to Old Testament story of Joseph, where Egypt experiences 7 yrs feast followed by 7 yrs famine.

→ Can crudely suggest: (see:)

- H s/t $0 < H < 1/2$

  ⇒ series movement is larger and "more random" than normal random movement; switching between hills valleys
  ⟺ Anti correlation / anti-persistent

- $H = 1/2$ random movement (CLT).
  diffusion

- $\frac{1}{2} < H < 1$, movement part of long term trend, persistent.

→ Can further parallel fractal/multi-fractal by:

uni-scaling:

$$\{E[(B_H(t+T) - B_H(t))^q]\}^{1/q} = \text{const}\ T^H$$

multi-scaling:

$$\{E[(B_H(t+T) - B_H(t))^q]\}^{1/q} \rightarrow \text{dependent on } q.$$

(multi-fractal)

→ Where does this story come from?

→ Statistical Hydrology (especially H. E. Hurst, 1880 - 1978)

→ see Mandelbrot / Wallis (1968)
motivated by work of H. E. Hurst.

→ realm: "Statistical Hydrology"

- how characterize precipitation / flooding patterns, with aim of reservoir construction?

- how account for "Noah", "Joseph" effects?

"Noah" ⇒ extreme precipitation very extreme indeed ⟺ large, rare events (Zipf, 1/f)
↓
concentration

"Joseph" ⇒ persistence (7 yrs feast, 7 yrs famine).
↓
memory / persistence

Claim: Brownian models cannot account for Noah, Joseph ⇒ underestimate complication of hydrological fluctuations, ⇒ make planning for reserves difficult.

Distribution of increments ??

⇒ Hurst examined annual discharge of Nile (i.e. flooding pattern), and observed empirically:

(ideal) reservoir capacity $\equiv R(\delta)$

standard deviation of discharges $\equiv S(\delta)$

$\delta \equiv$ # successive discharges

$$R(\delta)/S(\delta) \sim \delta^{H}$$

$.5 < H < 1$

defines R/S

'Hurst exponent' (excl. small s)

N.B.: $R(\delta)$ ⇒ pile 'energy' content
S ⇒ standard deviation discharges
δ ⇒ "time"

□

→ empirically, $\delta \sim .7 \rightarrow .85$, but $H \sim .5$ for diffusion ⇒ significant deviation.

persistence ⟺ Joseph effect

→ Memory.

→ Variance of series grows as $\sim \delta^H$.

→ $\frac{1}{2} < H < 1$ → "wild"

→ associated with persistence, long memory

⟺ intermittency.

→ '$1/f$' scaling of spectra associated with $\frac{1}{2} < H < 1$.

→ Meaning of H ⇔ general

→ H ⇒ index of dependence / index of long-range dependence

→ measures relative tendency of a time series to :

(a) – regress strongly to the mean (anti-persistent)
(b) – cluster in a direction (persistent)

i.e.

(a) H : 0 → 1/2

⇒ time series switching between high / low values (i.e. sticking to mean)

i.e. values high → low → high ....

with tendency persisting into future.

* ⇒ applicable to series for which autocorrelations at small lags can be + or –, but magn. autocorrelns. decay

⇒ "more random" than diffusion.

(b) $H = 1/2 \to 1$: – long term positive autocorreln of series

– suggests
high → high → high .....
series.
Persistent high value

i.e. {H is measure of memory in dynamics

How ? to H ?

Consider time series:

$X_1, X_2 \ldots X_n$

then H defined by:

$$C n^{H} = E\left[\frac{R(n)}{S(n)}\right]$$

(H ↑; E ↑ expectation)

$R(n) \equiv$ range of first n values.

$n \equiv$ # pts. (time span)

$S(n) \equiv$ standard deviation of first n values i.e. more quantitatively:

- estimate rescaled range on time dependence of observation,

i.e. is 'wild randomness' merely a masquerade of time varying Δ, etc.?

- N series:

divide into shorter series

$n = N, N/2, N/4,$

then rescaled range calculated for each n.

- n: $X_k, \ldots X_n$

1) mean $m = \frac{1}{n} \sum_{i=1}^{n} X_i$

2) adjust series to mean

$$Y_t = X_t - m, \quad t = 1, \ldots n$$

3) calculate cumulative deviate from mean

$$Z_t = \sum_{i=1}^{t} Y_i$$

4) Compute range R_j of deviate:

$$R(n) = \max(Z_1, \ldots Z_n) - \min(Z_1, \ldots Z_n)$$

5) Compute standard deviation

$$S(n) = \left(\frac{1}{n}\sum_{i=1}^{n}(X_i - m)^2\right)^{1/2}$$

then

$R(n)/S(n)$ gives H.

$H = \dfrac{\text{range of cumulative deviate}}{\text{standard deviation.}}$

Note: Avg. over all partial time series, n.

Can define generalized exponent:
derived from:

$H_q = H(q)$ → higher moment for time series $g(t)$

so, by analogy with turbulence structure function: time lag.

$$S_q = \langle |g(t+\tau) - g(t)|^q \rangle \sim \tau^{qH(q)}$$

obviously need average over $t > \tau$.

→ More:

- H related to fractal dimension D, where $1 < D < 2$, so

$$D = 2 - H$$

- spectral density B,

$$\langle B^2(\omega) \rangle \sim \omega^{-B}$$

$$B = 2H - 1$$

so $1/f \iff B = 1 \iff H = 1$

↓

divergence at low freq. (persistent)

$f^0 \iff B = 0 \iff H = 1/2$

↓

"white noise" (Brownian)

- see pics / Mandelbrot

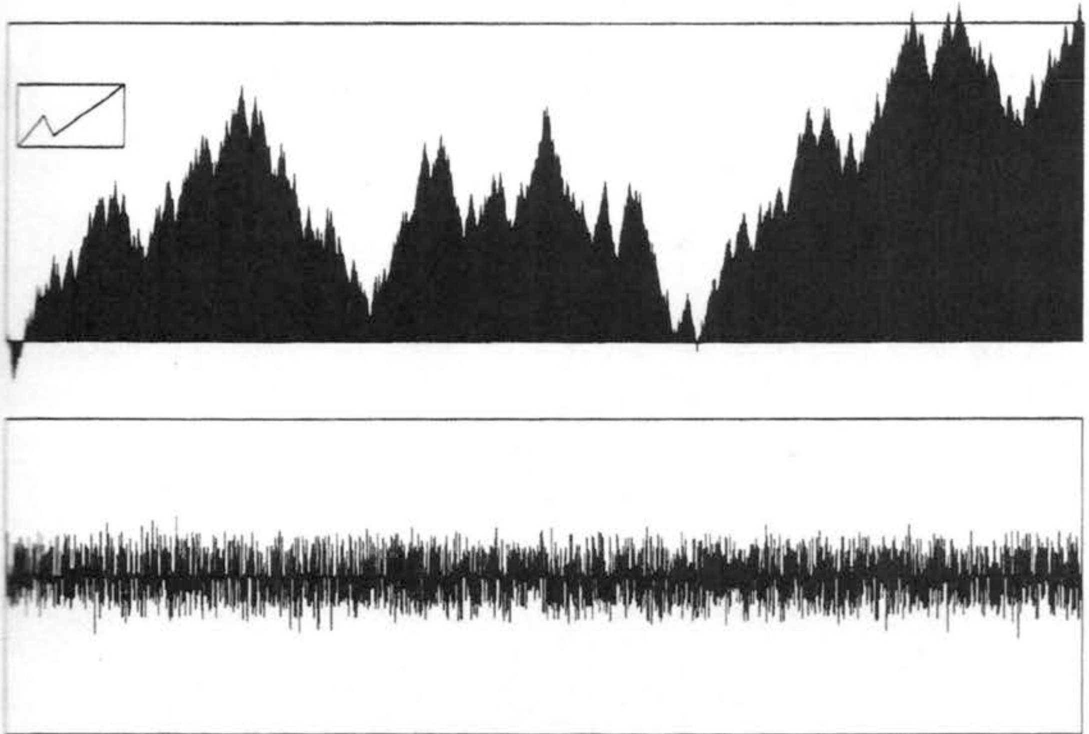

FIGURE E6-6. The top line illustrates a cartoon of Wiener Brownian motion carried to many recursion steps. The generator, shown in a small window, is identical to the generator A2 of Figure 2. At each step, the three intervals of the generator are shuffled at random; it follows that, after a few stages, no trace of a grid remains visible to the naked eye.

The second line shows the corresponding increments over successive small intervals of time. This is for all practical purposes a diagram of Gaussian "white noise" as shown in Figure 3 of Chapter E1.

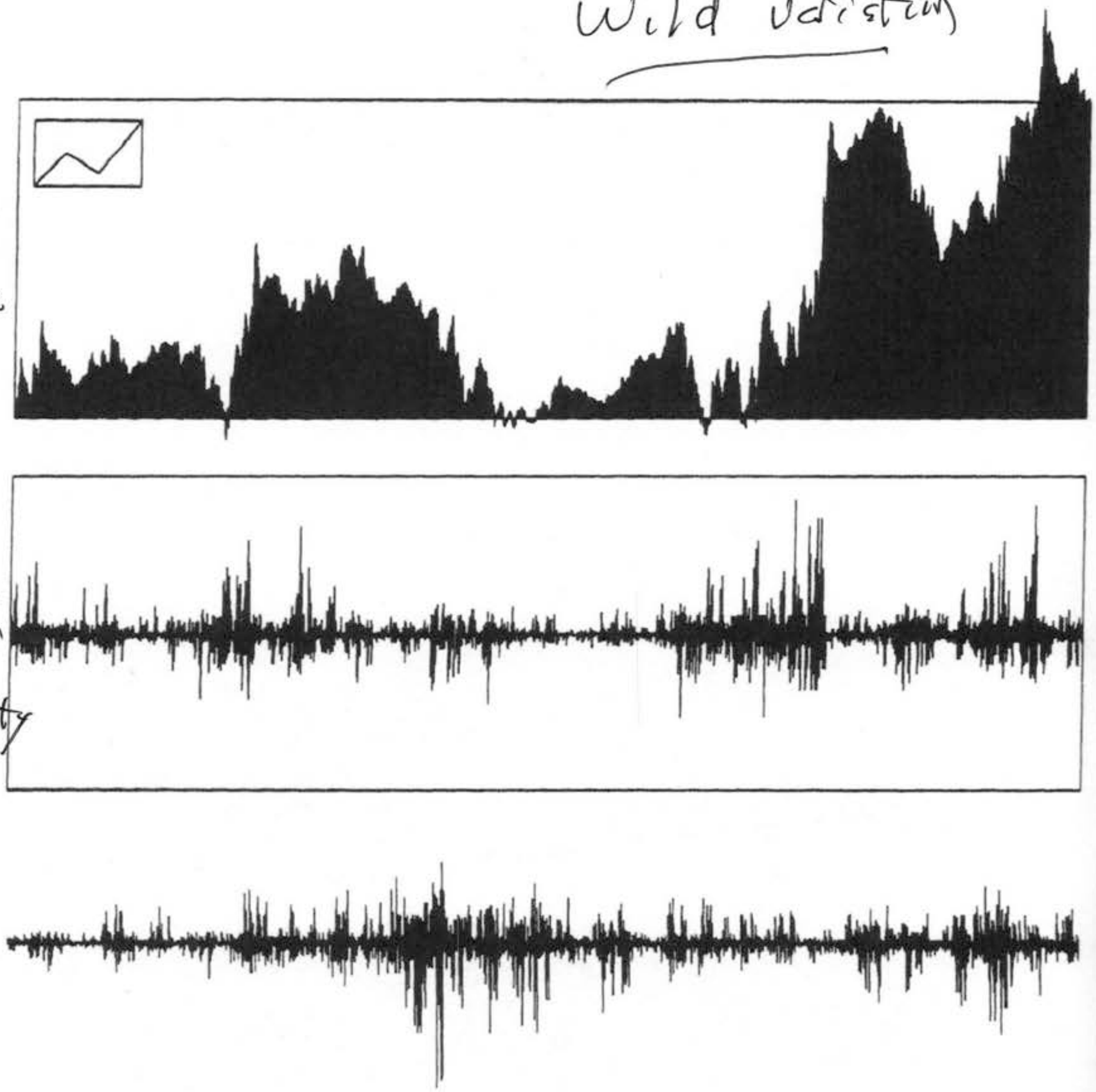

FIGURE E6-7. This figure reveals – at long last – the construction of Figure 2 of Chapter E1. The top line illustrates a cartoon of Wiener Brownian motion followed in a multifractal trading time. Starting with the three-box generator used in Figure 6, the box heights are preserved, so that D_T is left unchanged at $D_T = 2$ (a signature of Brownian motion), but the box widths are modified. (Unfortunately, the seed is not the same as in Figure 6.)

The middle line shows the corresponding increments. Very surprisingly, this sequence is a "white noise," but it is extremely far from being Gaussian. In fact, serial dependence is conspicuously high. The bottom line repeats the middle one, but with a different "pseudo-random" seed. The goal is to demonstrate once again the very high level of sample variability that is characteristic of wildly varying functions.

The resemblence to actual records exemplified by Figure 1 of Chapter E1 can be improved by "fine-tuning" the generator.

Gini ⟺ R/S ⟺ Kurtosis

→ How characterize "wild" randomness?
⇒ prob. distribution

- Levy flights are prime example of wild randomness

- Levy flight / Levy process } (pioneered by Paul Levy, coined - Mandelbrot)
is random walk in which ΔX distributed along $P(\Delta X)$ where $P(\Delta X)$ has "heavy" tail (→ power law).

e.g. Cauchy flight, $P(u) \sim A/1+u^2$

- Specific example: ~~[illegible]~~ probability

Consider $P(U) = P(U/u)$
↓
step size

$$P(U>u) = \begin{cases} 1 & : u<1 \\ u^{-\alpha} & : u \geq 1 \end{cases}$$
↓
power law

derived from Pareto distribution of incomes (power law).

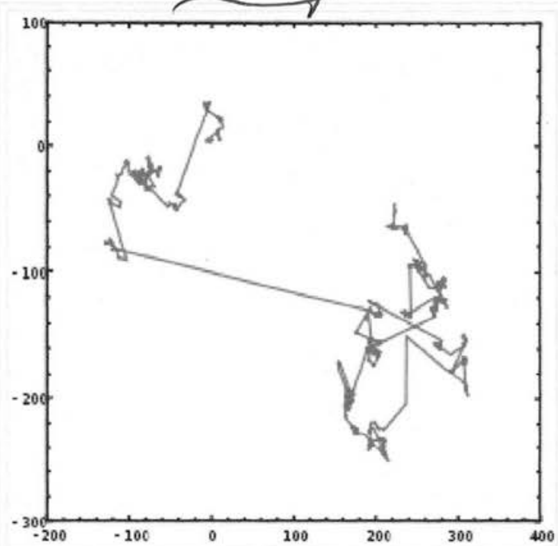

Figure 1. An example of 1000 steps of a Lévy flight in two dimensions. The origin of the motion is at [0,0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 1$ and $\beta = 0$ which is a Cauchy distribution. Note the presence of large jumps in location compared to the Brownian motion illustrated in Figure 2.

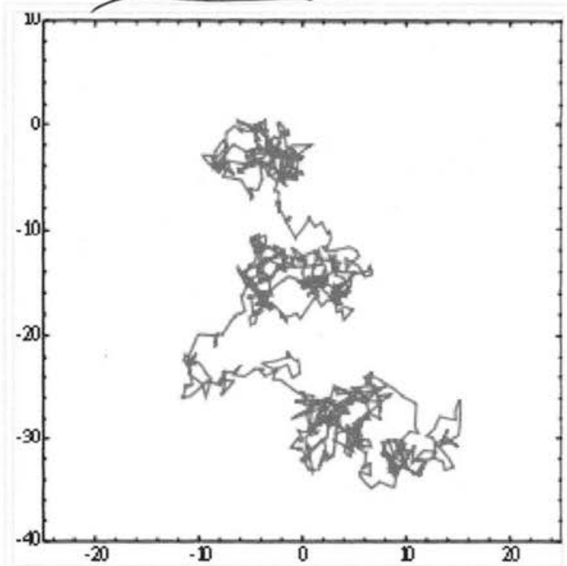

Figure 2. An example of 1000 steps of an approximation to a Brownian motion type of Lévy flight in two dimensions. The origin of the motion is at [0, 0], the angular direction is uniformly distributed and the step size is distributed according to a Lévy (i.e. stable) distribution with $\alpha = 2$ and $\beta = 0$ (*i.e.,* a normal distribution).

Applications

The definition of a Lévy flight stems from the mathematics related to chaos theory and is useful in stochastic measurement and simulations for random or pseudo-random natural phenomena. Examples include earthquake data analysis, financial mathematics, cryptography, signals analysis as well as many applications in astronomy, biology, and physics.

Another application is the Lévy flight foraging hypothesis. When sharks and other ocean predators can't find food, they abandon Brownian motion, the random motion seen in swirling gas molecules, for Lévy flight — a mix of long trajectories and short, random movements found in turbulent fluids. Researchers analyzed over 12 million movements recorded over 5,700 days in 55 data-logger-tagged animals from 14 ocean predator species in the Atlantic and Pacific Oceans, including silky sharks, yellowfin tuna, blue marlin and swordfish. The data showed that Lévy flights interspersed with Brownian motion can describe the animals' hunting patterns.[7][8][9][10] Birds and other animals[11] (including humans)[12] follow paths that have been modeled using Lévy flight (e.g. when searching for food).[13] Biological flight data can also apparently be mimicked by other models such as composite correlated random walks, which grow across scales to converge on optimal Lévy walks.[14] Composite Brownian walks can be finely tuned to theoretically optimal Lévy walks but they are not as efficient as Lévy search across most landscapes types, suggesting selection pressure for Lévy walk characteristics is more likely than multi-scaled normal diffusive patterns.[15]

More generally:
density

$$\begin{cases} P(U > u) = O(u^{-k}) \\ 1 < k < 3 \end{cases}$$

Brings us to:

OV: Pareto-Levy Law
Mandelbrot 1960.

→ emerged from economics, concerned with income distributions, especially tail.

→ Pareto (1897), Levy (1925)
— observed power law (P-L) ⇒ wild — flights Levy
— noted that P-L distribution satisfies a Limit Theorem (but not Gaussian)

→ Strong Pareto Law:

$P(u) \equiv$ % of indiv. with income $U > u$

$$P(u) = \begin{cases} (u/u_0)^{-\alpha}, & u > u_0 \\ 1, & u < u^0 \end{cases}$$

then density → $p(u) = -dP(u)/du$:
(pdf)

↦ Power Law.

$$p(u) = \begin{cases} \alpha(u_0) u^{-(\alpha+1)}, & u > u^0 \\ 0, & u < u^0 \end{cases}$$

p(u) characterized by { u_0 ⇒ scale factor; α ⇒ inequality index }

P(u) fits broad range of populations
(US tax payers, Renaissance towns etc)
(debated) (hint of robustness) ⇔ pdf is attractor in fctn space?

→ Weak Pareto Law " (more robust)

P(u) "behaves like" $(u/u_0)^{-\alpha}$, $u \to \infty$

⇒ { $p(u) \cong (u/u_0)^{-(\alpha+1)}$, ($\alpha < 2$ ⇒ fat tail)

need $\alpha > 2$ for 2nd moment convergence }

N.B. Competitors for Pareto:

— exponential tail: b ↦ ?

$$p(u) = k u^{-(\alpha+1)} e^{-bu}$$ $b \to 0$

- log-normals (why log normal relevant to incomes)
v.a

⇒ Thermodynamic Theories (?!)

- noting that Gaussian arises from Brownian motion ⇒ many small kicks in velocity,

ask

- can economic interactions exchange increments of money leading to P-L in equilibrium?
Is P-L result of a Limit Theorem?

⇒ NO! / Yes! large events dominate.

- P(u) decreases too slowly, large u.

- might try $\ln u \equiv v$ ⇒ leads to lognormal (can speak of additivity of $\log u$ increments, and convergence),

all debatable.

Percolation → {mild → wild transition ?! but

→ **Pareto - Levy Random Variables**

- Issue: Pareto law resilient to how income computed!

⇒

- Law emerges as a Limit Theorem.

i.e.

Levy Stable Distributions (attractors in fctn space)

- U_i → statistically indep. incomes (up to scale, origin)
- U', U'' follow P-L, then:

$U' \oplus U''$ follows law [illegible], where:

$\oplus$ = addition random variable

i.e.

$(a' U + b') \oplus (a'' U + b'') = aU + b$ → on PL

$a', a'' > 0$, $b', b'' > 0$ ⇒ $\exists\, a > 0, b$.

i.e. adding to P-L Law incomes ⇒ income "on" P-L Law.

∴ { P-L law is an example of an L-stable process! }

P-L densities 23.
α = 1.2, 1.5, 1.8

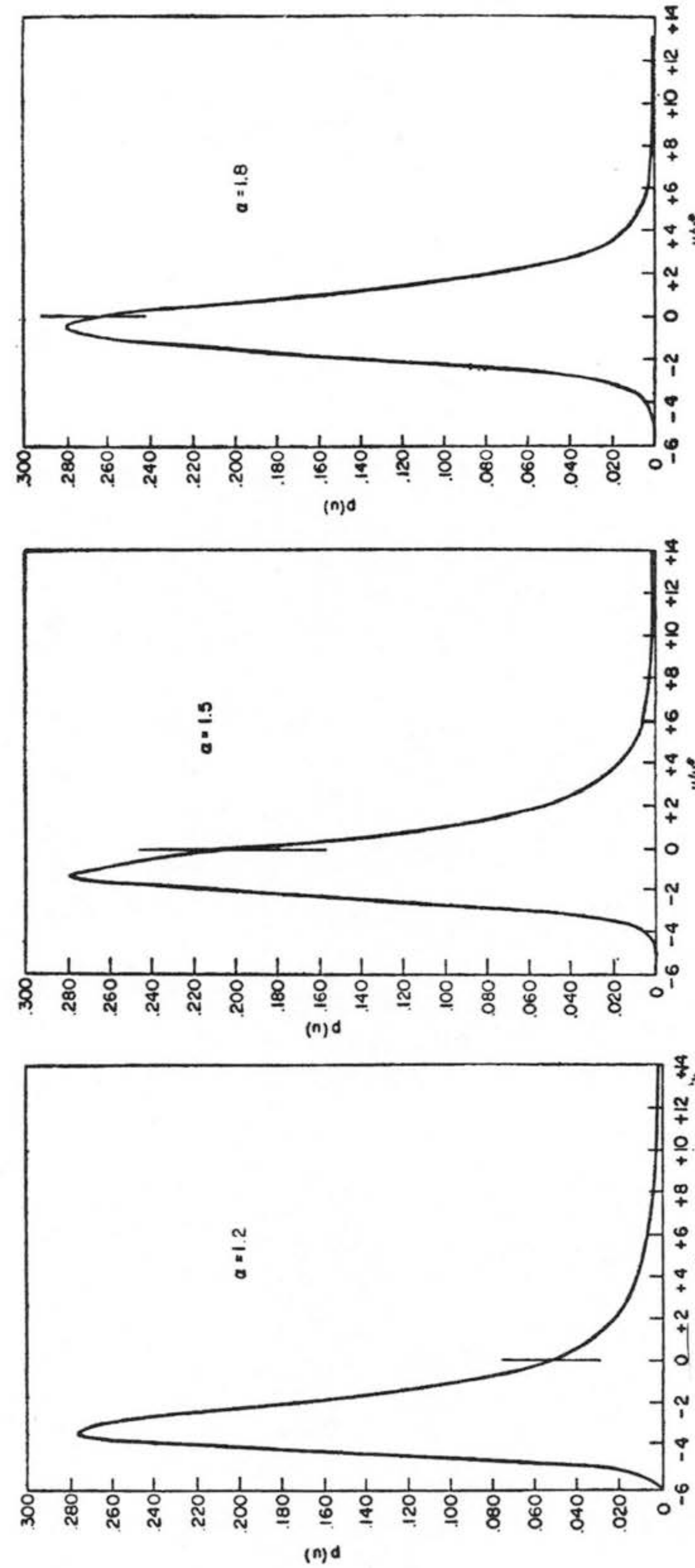

FIGURE 1: DENSITIES OF REDUCED P-L VARIABLES, ROF $M=0$ AND $\alpha=1.2,\ 1.5,\ 1.8$

→ Class of P-L-stable processes is those "stable", as above, under addition.

includes:

→ Gaussian (only stable distribution with finite variance!)

→ weak P-L laws with $1 < \alpha < 2$ (wild)

→ {only possible limit laws of weighted sums of identical and ~~indep~~ independent random variables

→ density p of P-L laws (see pics)

$$G(b) = \int_{-\infty}^{+\infty} e^{-bu} p(u)\, du$$

$$= \exp\left[(bu^*)^\alpha + Mb\right]$$

and Laplace transform.

{ α, u_*, $M \to E(u)$

→ Working Principle:

if ⇒ - sum of many components non-Gaussian
- skewed
- $E(u) < \infty$

} → reasonable assumption that follows P-L.

→ How derive pdf for Levy Flights with "Wild Distribution"?

⇒ Fractional Kinetics / { case of "anomalous" diffusion

i.e.

$$\frac{\partial P}{\partial t} = -\partial_x \left(F(x,t)\, P(x,t) \right) + \gamma \frac{\partial^\alpha P(x,t)}{\partial |x|^\alpha}$$

$\alpha \neq 2$ ⇐ anomaly

where:

$$k^\alpha \mathcal{F}[P] = \mathcal{F}\left[\partial^\alpha P / \partial |x|^\alpha \right]$$

↑ Fourier transform

defines fractional derivative.

N.B.: – pinch possible
– γ, α parameters.